J. Fluid Mech. (1975), *vol.* 67, *part* 4, *pp.* 743–761
Printed in Great Britain

On the viscous flow about the trailing edge of a rapidly oscillating plate

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(Received 29 October 1973 and in revised form 28 June 1974)

The incompressible laminar flow in the neighbourhood of the trailing edge of an aerofoil undergoing sinusoidal oscillations of high frequency and low amplitude in a uniform stream is described in the limit as the Reynolds number R tends to infinity. The aerofoil is replaced by a flat plate on the assumption that leading-edge stall does not take place. It is shown that, for oscillations of non-dimensional frequency $O(R^{\frac{1}{4}})$ and amplitude $O(R^{-\frac{9}{16}})$, a rational description of the flow at the trailing edge is based on a subdivision of the boundary layer above the plate into five distinct regions. Asymptotic analytic solutions are found in four of these, whilst in the fifth a linearized solution yields an estimate for the viscous correction to the circulation determined by the Kutta condition.

1. Introduction

The steady flow of an incompressible viscous fluid near the trailing edge of a flat plate aligned with a uniform stream has been studied by both Stewartson (1969) and Messiter (1970). When the Reynolds number R is large the flow in the neighbourhood of the trailing edge has a three-layer or triple-deck structure. The reason for the presence of this triple deck is the discontinuity in the boundary condition at the trailing edge: that of zero tangential velocity on the plate is replaced by that of zero stress on the centre-line of the wake. This work was extended to the case of an aerofoil at incidence by Brown & Stewartson (1970, hereafter referred to as I) in the situation when the angle of incidence is $O(R^{-\frac{1}{16}})$. It was argued that for larger angles the flow would separate before the trailing edge was reached, for smaller angles the flow would be a small perturbation of the unseparated zero-incidence case, but that this critical size of angle resulted in an adverse pressure gradient on the upper side of the plate which was of the same order of magnitude as the favourable pressure gradient induced by the triple deck. The flow therefore would separate in the immediate neighbourhood of the trailing edge, a phenomenon interpreted as trailing-edge stall. The arbitrary constant in the outer flow, which determines the circulation around the aerofoil, was found to be a function of the Reynolds number. When the Reynolds number is infinite the value of this arbitrary constant is determined by the usual Kutta condition of finite velocity (and hence zero loading) at the trailing edge. The match between the triple deck and the outer flow yielded a correction to this value which was of relative order $R^{-\frac{3}{8}}$. It was then possible to calculate the contribution of this correction to the lift coefficient.

In steady flow the Kutta condition may be regarded as well understood, but in unsteady flow the situation is not so clear. Hancock (1973, see Riley 1974, p. 35), for example, suggested that, in general, there are two independent conditions (one states that the loading at the trailing edge must be zero, the other that the inviscid flow must separate from the trailing edge), and argued that both these may be satisfied by a suitable choice of *two* arbitrary constants appearing in the solution. Van de Vooren & Van de Vel (1964) chose their single arbitrary constant to ensure zero loading in the case of an aerofoil with zero trailing-edge angle though if the angle was non-zero this was not possible with their model as only the stronger of two singularities could be removed. For the problem of the mixing of a uniform stream in the region above the plate with static fluid below Orszag & Crow (1970) introduced the notion of three alternative conditions to render the flow unique. They found the analytic form of the time-harmonic, spatially undulating vortex sheet shed from the trailing edge in the situations in which (i) the extreme positions of the sheet form a symmetrically disposed parabola, (ii) the fluid from the upper side of the plate does not have to turn through an angle greater than π, and (iii) the vortex sheet leaves the plate tangentially at all times. The authors rather favoured (ii) as being most realistic physically though recent experiments by Pfizenmaier & Bechert (1973) on the exit condition for alternating flow at the trailing edge of a nozzle indicate that it is (i) which occurs in practice. It is possible that a viscous study would resolve the dilemma of which theoretical condition should be applied.

In the present paper we do not attempt to consider the more difficult problem of Orszag & Crow but discuss a situation which extends the theory in I to the case of an aerofoil in unsteady motion. For simplicity we take this to be a flat plate of length l which is held fixed at its mid-point and performs a sinusoidal pitching motion of small amplitude $\alpha^* l$ and high frequency ω^*. The justification for considering a flat-plate aerofoil, so that the flow remains unseparated until it enters the trailing-edge region, requires certain restrictions on the thickness of the aerofoil and is discussed in I. The parameters of the problem are the Reynolds number $R = U_\infty l/\nu$, where U_∞ is the mainstream speed and ν is the kinematic viscosity, the non-dimensional amplitude α^* and the frequency parameter $S = \omega^* l/U_\infty$. The Reynolds number is assumed to be large and the orders of magnitude of the other two parameters are chosen in terms of R. The outer potential flow contains an arbitrary constant which will, as in the steady case of I, be Reynolds number dependent. We choose the limiting value of this constant as $R \to \infty$ so that the loading at the trailing edge vanishes at all times, and verify that this is indeed a consistent inviscid limit by finding the viscous correction to this value. This is achieved by matching the outer flow to that in a triple deck of streamwise extent $O(lR^{-\frac{3}{8}})$ centred on the trailing edge, the structure of which is essentially that of Stewartson (1969). The value of this constant determines the circulation around the aerofoil and we are thus able to calculate the viscous correction to the lift and moment.

The orders of magnitude of α^* and S are chosen as follows. The flow is to enter a triple deck of streamwise extent $O(lR^{-\frac{3}{8}})$ centred on the trailing edge, the structure of which is to be essentially that of Stewartson (1969). The main deck of this triple deck is of thickness $O(lR^{-\frac{1}{2}})$ and the lower deck of thickness $O(lR^{-\frac{5}{8}})$. Upstream of the triple deck the flow will be a perturbation to that of Blasius (1908) and since ω^* is large there will be a Stokes layer in the neighbourhood of the wall of thickness $O((\nu/\omega^*)^{\frac{1}{2}})$. In this layer the viscous term in the equation of motion is balanced by the term involving the time derivative and in order that the solution will match with that in the lower deck, we choose the order of magnitude of S so that the two layers have the same thickness: thus $S = O(R^{\frac{1}{4}})$. If the order of magnitude of S is smaller than $O(R^{\frac{1}{4}})$ the flow will be merely a perturbation of that for a steady aerofoil at incidence, and if it is larger it is probable that the triple-layer flow near the trailing edge will be destroyed by the rapid oscillation. Once the order of magnitude of S has been determined, that of α^* follows in exactly the same way as did that of the angle of incidence for the steady lifting aerofoil. The favourable pressure gradient induced by the triple deck is to be of the same order of magnitude as the adverse pressure gradient induced by the oscillation. It emerges that for this to hold we must have $\alpha^* = O(R^{-\frac{9}{16}})$.

The plan of the paper is as follows. In §2 we obtain the potential solution and discuss the form of the Kutta condition. Subsequent sections are concerned with the perturbed Blasius flow and its Stokes layer, and with an additional two-layer deck which did not occur in the steady problem which we term the fore deck. This is of streamwise extent $O(lR^{-\frac{1}{4}})$ and lies between the perturbed Blasius flow and the Stewartson triple deck. The boundary layers on the two sides of the plate then separately enter the triple deck and the flow in the lower deck is governed by partial differential equations similar to, though more complicated than, those which occur in the steady case. No numerical solution of these equations has been undertaken though it is shown that they possess the correct asymptotic form both upstream, where a match with the fore deck is required, and downstream, where a match is necessary with a Goldstein (1930) wake solution with the centre-line displaced. The various regions of the flow are illustrated in figure 1. Region I is the potential flow and regions II_1 and II_2 are the perturbed Blasius layer and its Stokes layer respectively. The two layers of the fore deck are denoted by III_1 and III_2 while region IV is divided into the three layers of the triple deck. Regions V comprise the outer and inner modified Goldstein wakes.

Although no complete solution of the lower-deck equations is available, upstream of the trailing edge it is reasonable to linearize about the uniform shear with which the streamwise velocity must match at the outer edge. A solution of the resulting equations for the difference in the streamwise velocity components on the two sides of the plate is then obtainable by Wiener–Hopf arguments of the type used in I for the steady case. This is undertaken in §7. It avoids solving for the boundary layer in the wake, and also yields a solution for the antisymmetric part of the pressure, which must vanish downstream of the trailing edge. The final result is an estimate of the time-dependent viscosity correction to the circulation term given by the Kutta condition, as assumed in §2. This leads to

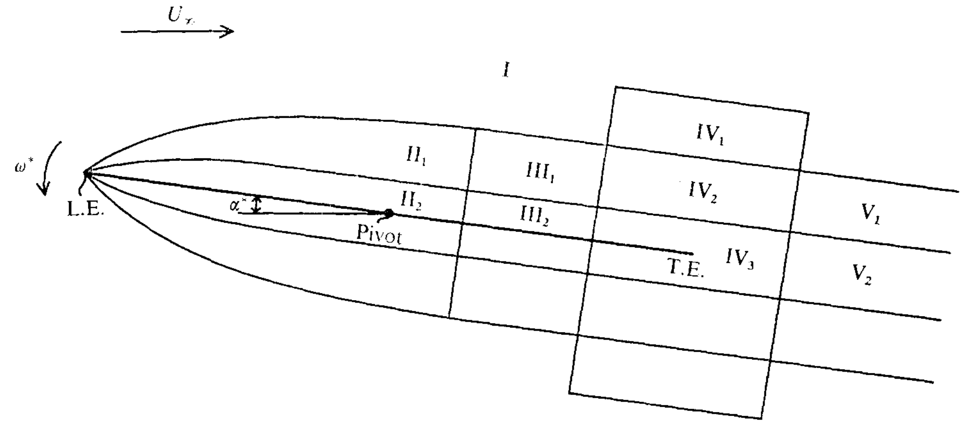

FIGURE 1. The regions of flow in the neighbourhood of the trailing edge on the upper side of the plate (not to scale). I, potential flow; II, perturbed Blasius flow and inner Stokes layer; III, the fore deck; IV, the triple deck; V, modified Goldstein wake.

corrections to the lift and moment which have a phase lag of $\frac{1}{4}\pi$ behind the leading-order terms. It emerges that there is a stagnation point of the outer flow at a distance $O(lR^{-\frac{7}{8}})$ from the trailing edge which moves from one side of the plate to the other with a phase lag of $\frac{1}{4}\pi$ relative to the oscillation of the aerofoil.

In the penultimate section we note the modifications required if the aerofoil oscillates in a plunging rather than a pitching mode. The arguments are unaltered for an appropriate relationship between the respective amplitudes, though the expressions for the lift and moment are different in form.

2. The external potential flow

Consider a flat plate of length l with mid-point at the origin O of a set of Cartesian co-ordinates (x^*, y^*) fixed in space. The plate performs oscillations of amplitude $\alpha^* l$ and frequency ω^* in an incompressible fluid of constant density ρ which has uniform velocity U_∞ at infinity. At any time t^*, the equation of the plate is thus

$$y^* = -2\alpha^* x^* \exp(i\omega^* t^*) \quad (-\tfrac{1}{2}l \leqslant x^* \leqslant \tfrac{1}{2}l). \tag{2.1}$$

If terms $O(\alpha^{*2})$ are neglected the potential flow in region I of figure 1 may be obtained by thin aerofoil theory and the solution is given by Robinson & Laurmann (1956, chap. 5). The principal result we shall need is that the pressure on the upper surface of the plate is given by

$$\frac{p^*-p_\infty}{\rho U_\infty^2} = \tfrac{1}{2}a_0(t^*)\left(\frac{l-2x^*}{l+2x^*}\right)^{\frac{1}{2}} + \tfrac{1}{2}B(t^*)\left(\frac{l+2x^*}{l-2x^*}\right)^{\frac{1}{2}} + a_1(t^*)\left(1-\frac{4x^{*2}}{l^2}\right)^{\frac{1}{2}} + 4a_2(t^*)\frac{x^*}{l}\left(1-\frac{4x^{*2}}{l^2}\right)^{\frac{1}{2}} \quad (-\tfrac{1}{2}l \leqslant x^* \leqslant \tfrac{1}{2}l), \tag{2.2}$$

where p_∞ is the pressure at infinity. The term $a_0(t^*)$ represents a leading-edge singularity, whilst the term $B(t^*)$ is undetermined by the inviscid analysis and would be set equal to zero by the Kutta–Joukowski hypothesis of finite pressure at the trailing edge. The functions $a_1(t^*)$ and $a_2(t^*)$ are determined as

$$a_1(t^*) = -2i\alpha^*\frac{\omega^* l}{U_\infty}\exp(i\omega^* t^*), \quad a_2(t^*) = \tfrac{1}{8}\alpha^*\left(\frac{\omega^* l}{U_\infty}\right)^2\exp(i\omega^* t^*), \tag{2.3}$$

and although $a_0(t^*)$ may be written in terms of the Theodorsen function we shall only need its value for large $\omega^* l/U_\infty$, which is

$$a_0(t^*) = \tfrac{1}{2}i\alpha^*(\omega^* l/U_\infty)\exp(i\omega^* t^*) + O(\alpha^*). \tag{2.4}$$

We aim to determine $B(t^*)$ by taking into account the viscous effects in the immediate neighbourhood of the trailing edge and write

$$B(t^*) = \alpha^*(\omega^* l/U_\infty)^2 C\exp(i\omega^* t^*), \tag{2.5}$$

where C is an unspecified complex constant; in the following section we shall make an assumption regarding the order of magnitude of $|C|$ which will be justified *a posteriori*. The pressure on the lower side of the plate is obtained by changing the sign of α^* in the above expressions. The total lift L and pitching moment M about the mid-chord of the plate due to the potential flow are easily obtained from (2.2). We find that

$$\left.\begin{aligned} L &= -\tfrac{1}{2}\pi\rho l U_\infty^2\{a_0(t^*) + a_1(t^*) + B(t^*)\},\\ M &= \tfrac{1}{8}\pi\rho l^2 U_\infty^2\{a_0(t^*) - a_2(t^*) - B(t^*)\},\end{aligned}\right\} \tag{2.6}$$

where the real part of these and all complex expressions is to be taken.

The velocity components u^* and v^* of this potential flow will also be required, and since our interest is centred on the trailing edge of the plate, where

$$\tfrac{1}{2} - x^*/l \ll 1,$$

and the frequency parameter $S = \omega^* l/U_\infty$ is assumed to be large, we replace (2.2) by

$$\frac{p^* - p_\infty}{\rho U_\infty^2} = \tfrac{1}{2}\alpha^* S^2\left\{\left(\frac{1}{2} - \frac{x^*}{l}\right)^{\frac{1}{2}} + C\left(\frac{1}{2} - \frac{x^*}{l}\right)^{-\frac{1}{2}}\right\}\exp(i\omega^* t^*). \tag{2.7}$$

The streamwise velocity u^* evaluated on the plate will be the mainstream for the boundary-layer discussion of the following section. If we denote it by $U_1(x^*, t^*)$ then U_1 satisfies Bernoulli's equation

$$\frac{\partial U_1}{\partial t^*} + U_\infty\frac{\partial U_1}{\partial x^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*}, \tag{2.8}$$

where p^* is given by (2.7). The general solution of this can easily be obtained but we shall find it more convenient to solve it in the particular limiting situations to be considered. A similar statement applies to the velocity v^* evaluated on the centre-line of the wake ($y^* = 0, x^* \geqslant \tfrac{1}{2}l$), whose value $V_1(x^*, t^*)$ satisfies the equation

$$\frac{\partial V_1}{\partial t^*} + U_\infty\frac{\partial V_1}{\partial x^*} = -U_\infty^2\frac{\alpha^*}{4l}S^2\left\{\left(\frac{x^*}{l} - \frac{1}{2}\right)^{-\frac{1}{2}} + C\left(\frac{x^*}{l} - \frac{1}{2}\right)^{-\frac{3}{2}}\right\}\exp(i\omega^* t^*), \tag{2.9}$$

for $x^*/l - \tfrac{1}{2} \ll 1$.

It is convenient at this stage to specify the orders of magnitude of the parameters involved. We define ϵ by

$$\epsilon^{-8} = R = U_\infty l/\nu, \tag{2.10}$$

and stipulate that $\epsilon \ll 1$. Then the streamwise extent of the Stewartson triple deck is $O(\epsilon^3)$ and those of the outer, main and lower decks are $O(\epsilon^3)$, $O(\epsilon^4)$ and $O(\epsilon^5)$ respectively. The pressure induced by the triple deck is $O(\epsilon^2)$. As indicated in §1 the orders of magnitude of the parameters occurring in this unsteady flow will be chosen so that the fluid experiences a deceleration owing to the oscillation (on the appropriate side of the plate at any time) which is comparable with the acceleration induced by the triple deck. We first consider the frequency parameter S, which we have already assumed to be large. Upstream of the triple deck the boundary layer on each side of the plate will be a perturbation of that of Blasius together with a Stokes layer in the neighbourhood of the wall of thickness $O((\nu/\omega^*)^{\frac{1}{2}})$. If we specify the order of magnitude of S so that the thicknesses of the Stokes layer and the lower deck are of the same order of magnitude it follows that

$$S = O(\epsilon^{-2}). \tag{2.11}$$

The appropriate order of magnitude of α^* now follows from (2.7). We require that in the triple deck the pressure induced by the change of boundary condition at the trailing edge, and the perturbation pressure caused by the oscillation be comparable. Thus when $\frac{1}{2} - x^*/l$ is $O(\epsilon^3)$, then $(p^* - p_\infty)/\rho U_\infty^2$, as given by (2.7), is to be $O(\epsilon^2)$. Thus

$$\alpha^* S^2 = O(\epsilon^{\frac{1}{2}}), \tag{2.12}$$

as long as the constant $|C|$ is not of larger order than ϵ^3. We now make the assumption that $|C|$ is $O(\epsilon^3)$ so that the two terms of (2.7) are then of the same order when $\frac{1}{2} - x^*/l$ is $O(\epsilon^3)$. The value of C will be determined by the match between the lower deck and the flow upstream of the trailing edge. In §7 we obtain an estimate of this value, and show that with the above choices of the orders of magnitude of S and α^* the assumption regarding the order of magnitude of $|C|$ is indeed consistent.

3. The perturbed Blasius boundary layer

To study the flow in the boundary layer and triple deck it is convenient to refer the motion to axes which are fixed in the plate and have their origin at the trailing edge. We therefore define co-ordinates (x, y) by

$$x^*/l - \tfrac{1}{2} = x + \alpha^* y e^{it}, \quad y^*/l = y - \alpha^*(x + \tfrac{1}{2}) e^{it}, \tag{3.1}$$

and corresponding velocities (u, v) by

$$\left.\begin{aligned} u^* - \alpha^* v^* \exp(i\omega^* t^*) &= U_\infty(u + i\alpha^* S y e^{it}), \\ \alpha^* u^* \exp(i\omega^* t^*) + v^* &= U_\infty[v - i\alpha^* S(x + \tfrac{1}{2}) e^{it}], \end{aligned}\right\} \tag{3.2}$$

where $t = \omega^* t^*$ and terms $O(\alpha^{*2})$ have been neglected in accordance with the thin aerofoil theory assumption of §2. The equation of the plate is now

$$y = 0 \quad (-1 \leqslant x \leqslant 0)$$

for all t.

The Navier–Stokes equations are modified by additional terms to account for the rotation and are given in full by Shen & Crimi (1965). It emerges that for the accuracy of the solution presented here these extra terms are of lower order of magnitude and need not be considered in any of the flow regions.

The boundary-layer flow that approaches the trailing edge of the plate is that of Blasius together with a perturbation that is $O(\alpha^*)$. The mainstream for this boundary layer is $u = 1 + U(x)\,e^{it}$, where the equation satisfied by $U(x)$ is obtained from (2.8), (3.1) and (3.2) as

$$(iS_0/\epsilon^2)\,U(x) + U'(x) = \tfrac{1}{4}\epsilon^{\frac{1}{2}}\alpha_0 S_0^2\{(-x)^{-\frac{1}{2}} - \epsilon^3 C_0(-x)^{-\frac{3}{2}}\}, \tag{3.3}$$

in which the terms neglected are again $O(\alpha^{*2})$. Here we have written

$$S = S_0/\epsilon^2, \quad \alpha^* = \epsilon^{\frac{9}{2}}\alpha_0, \quad C = \epsilon^3 C_0, \tag{3.4}$$

and henceforth regard S_0, α_0 and C_0 as independent of ϵ.

The perturbed Blasius flow occurs in a region upstream of the trailing edge in which $-x = O(1)$ and $y = O(\epsilon^4)$; this is region II_1 of figure 1. It follows from (3.3) that the mainstream for this flow is

$$1 - \tfrac{1}{4}\epsilon^{\frac{5}{2}} i\alpha_0 S_0(-x)^{-\frac{1}{2}}\,e^{it}. \tag{3.5}$$

It is also clear from (3.3) that (3.5) will be inadequate in a region in which $-x = O(\epsilon^2)$, which we term the fore deck and consider in the following section, and in the triple deck, in which $-x = O(\epsilon^3)$. If we write

$$\bar{y} = y/\epsilon^4, \quad \bar{v} = v/\epsilon^4, \tag{3.6}$$

the boundary-layer equations appropriate to this mainstream are

$$\frac{\partial u}{\partial x} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \quad \frac{S_0}{\epsilon^2}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \bar{v}\frac{\partial u}{\partial \bar{y}} = \tfrac{1}{4}\epsilon^{\frac{1}{2}}\alpha_0 S_0^2(-x)^{-\frac{1}{2}}e^{it} + \frac{\partial^2 u}{\partial \bar{y}^2}. \tag{3.7}$$

For $\epsilon \ll 1$ we write $(u, \bar{v})$ in (3.7) as a perturbation to the solution of Blasius, and apply the condition that u tends to the mainstream velocity of (3.5) as $\bar{y} \to \infty$. We obtain

$$u = f_B'(\zeta) - \tfrac{1}{4}\epsilon^{\frac{5}{2}} i\alpha_0 S_0(-x)^{-\frac{1}{2}}e^{it} + O(\epsilon^{\frac{9}{2}}), \tag{3.8}$$

$$\bar{v} = -(1+x)^{-\frac{1}{2}}(f_B - \zeta f_B') + \tfrac{1}{8}\epsilon^{\frac{5}{2}} i\alpha_0 S_0\,\bar{y}(-x)^{-\frac{3}{2}}e^{it} + O(\epsilon^{\frac{9}{2}}), \tag{3.9}$$

where $\zeta = \bar{y}/(1+x)^{\frac{1}{2}}$ and $f_B(\zeta)$ is the Blasius function with

$$f_B(0) = f_B'(0) = 0 \quad \text{and} \quad f_B''(0) = \lambda = 0{\cdot}3321.$$

The term $O(\epsilon^{\frac{9}{2}})$ in (3.8) will be affected by the terms in the Navier–Stokes equation due to the rotating axes. Clearly (3.8) does not satisfy the no-slip condition on the plate, but this is accomplished by consideration of an inner Stokes layer of thickness $O(\epsilon^5 l)$, which is region II_2 of figure 1. In this region $\bar{y} = \epsilon z$ and the solution which matches with (3.8) and (3.9) is easily found to be

$$u = \frac{\lambda\epsilon z}{(1+x)^{\frac{1}{2}}} - \tfrac{1}{4}\epsilon^{\frac{5}{2}}\frac{i\alpha_0 S_0}{(-x)^{\frac{1}{2}}}[1 - \exp(-i^{\frac{1}{2}}S_0^{\frac{1}{2}}z)]\,e^{it} + O(\epsilon^{\frac{9}{2}}), \tag{3.10}$$

$$\bar{v} = \frac{\lambda\epsilon^2 z^2}{4(1+x)^{\frac{3}{2}}} + \tfrac{1}{8}\epsilon^{\frac{7}{2}}\frac{i\alpha_0 S_0}{(-x)^{\frac{3}{2}}}\left(z - \frac{1}{i^{\frac{1}{2}}S_0^{\frac{1}{2}}}[1 - \exp(-i^{\frac{1}{2}}S_0^{\frac{1}{2}}z)]\right)e^{it}, \tag{3.11}$$

where $i^{\frac{1}{2}} = (1+i)/2^{\frac{1}{2}}$.

4. The fore deck

It emerges that the solution (3.8) in the main part of the boundary layer cannot be matched to the main deck of the triple deck, in which $x = O(\epsilon^3)$. The reason for this is immediately obvious on examination of (3.3), which indicates the existence of a region with $x = O(\epsilon^2)$ in which the neglect of the term $U'(x)$ corresponding to the solution (3.5) is not justified. The required transition is supplied by the fore deck, denoted by III in figure 1, and in it we write

$$x_1 = x/\epsilon^2 = O(1). \tag{4.1}$$

The fore deck itself consists of two decks, a main deck III_1 where $\bar{y} = O(1)$ and a lower deck III_2 where $z = O(1)$. The latter is effectively redundant since the sublayer solution (3.10) and (3.11) remains undisturbed and matches with the lower deck of the triple deck. The pressure, which to leading order is independent of y, remains unaffected throughout the fore deck, and is thus equal to the inviscid pressure, which from (3.3) is

$$p = (p^* - p_\infty)/\rho U_\infty^2 = \tfrac{1}{2}\epsilon^{\frac{3}{2}}\alpha_0 S_0^{\frac{3}{2}}(-x_1)^{\frac{1}{2}} e^{it} + \dots . \tag{4.2}$$

The outer flow for this region is obtained from (3.3) as

$$1 + \tfrac{1}{4}\epsilon^{\frac{3}{2}}\alpha_0 S_0^{\frac{3}{2}} \exp(it - iS_0 x_1) \int_{-\infty}^{x_1} (-x_1')^{-\frac{1}{2}} \exp(iS_0 x_1')\, dx_1'. \tag{4.3}$$

In the main deck of the fore deck we write

$$\left.\begin{aligned} u &= U_0(\bar{y}) + \epsilon^{\frac{3}{2}} U_2(x_1, \bar{y})\, e^{it} + \dots, \\ v &= \qquad\quad\ \epsilon^{\frac{3}{2}} V_2(x_1, \bar{y})\, e^{it} + \dots, \end{aligned}\right\} \tag{4.4}$$

and substitute into the full equations of motion in the rotating co-ordinates. Here $U_0(\bar{y}) = f_B'(\bar{y})$ and is the Blasius profile evaluated at the trailing edge. We use the known pressure (4.2) and eliminate U_2 by means of the continuity equation to obtain the following equation for V_2:

$$iS_0 \frac{\partial V_2}{\partial \bar{y}} + U_0(\bar{y}) \frac{\partial^2 V_2}{\partial x_1 \partial \bar{y}} - U_0'(\bar{y}) \frac{\partial V_2}{\partial x_1} = -\tfrac{1}{8}\alpha_0 S_0^{\frac{3}{2}}(-x_1)^{-\frac{3}{2}}. \tag{4.5}$$

Now as $x_1 \to -\infty$ equations (4.4) must match with the normal velocity in the main part of the boundary layer upstream, so that from (3.9) we require

$$V_2(x_1, \bar{y}) \sim i\alpha_0 S_0 \bar{y}/8(-x_1)^{\frac{3}{2}} \quad \text{as} \quad x_1 \to -\infty. \tag{4.6}$$

Also, as $\bar{y} \to \infty$, u, as given by (4.4), must tend to the expression in (4.3). The condition obtained therefrom on $\partial U_2/\partial x_1$ leads to a condition on $\partial V_2/\partial \bar{y}$. A series of substitutions gives the solution of (4.5) satisfying this condition and (4.6) as

$$V_2(x_1, \bar{y}) = \frac{U_0(\bar{y})}{g_0'(\bar{y})}\left(\frac{\partial w}{\partial x_1} + g_0 w - W_0\right), \tag{4.7}$$

where

$$w(x_1,\bar{y}) = g_0'(\bar{y})\int_{y'=0}^{y'=\bar{y}} \frac{\exp(-g_0(y')\,x_1)}{g_0'(y')}\,dy' \int_{x'=-\infty}^{x'=x_1} H(x',y')\exp(g_0(y')x')\,dx' + m(x_1), \tag{4.8}$$

$$H(x_1,\bar{y}) = \frac{\partial W_0}{\partial \bar{y}} - \frac{g_0''}{g_0'}W_0, \tag{4.9}$$

$$W_0(x_1,\bar{y}) = -\alpha_0 S_0^2/8U_0^2(\bar{y})\,(-x_1)^{\frac{3}{2}}, \quad g_0(\bar{y}) = iS_0/U_0(\bar{y}), \tag{4.10}$$

and $m(x_1)$ satisfies $m(x_1)\,(-x_1)^{\frac{3}{2}} \to 0$ as $x_1 \to -\infty$, but is otherwise arbitrary.

In the limit as $\bar{y} \to \infty$ the solution is consistent with the mainstream velocity (4.3). The function $m(x_1)$ is determined by the match with the normal velocity in the lower deck as $\bar{y} \to 0$, which is given by the sublayer solution (3.11) written in terms of the co-ordinate x_1. An investigation of the solution (4.7) shows that

$$V_2(x_1,\bar{y}) \sim iS_0 m(x_1) + i\bar{y}\alpha_0 S_0/8(-x_1)^{\frac{3}{2}} \quad (\bar{y} \to 0), \tag{4.11}$$

so that (4.4) matches with (3.11) only if $m(x_1) = 0$. We shall also require the structure of the solution (4.7) as $x_1 \to 0-$; here we find that

$$\left.\begin{aligned} V_2(x_1,\bar{y}) &\sim i\alpha_0 S_0 U_0(\bar{y})/8\lambda(-x_1)^{\frac{3}{2}} \\ U_2(x_1,\bar{y}) &\sim -i\alpha_0 S_0 U_0'(\bar{y})/4\lambda(-x_1)^{\frac{1}{2}} \end{aligned}\right\} \quad (x_1 \to 0-), \tag{4.12}$$

which are of the familiar form for a match with a Stewartson triple deck.

5. The triple deck

On leaving the fore deck, the flow enters the triple deck (region IV of figure 1) centred at the trailing edge of the plate, where

$$x_2 = x/\epsilon^3 = O(1). \tag{5.1}$$

As $x_2 \to -\infty$, the solutions in the lower and main decks of the triple deck, where $z = O(1)$ and $\bar{y} = O(1)$ respectively, match with the solutions in the corresponding regions of the fore deck whilst the solution in the upper deck, where

$$Y = y/\epsilon^3 = O(1),$$

matches with the potential solution (4.3).

We first consider the main deck, where

$$p = \epsilon^2 p_m(x_2,\bar{y},t) + \ldots, \tag{5.2}$$

$$u = U_0(\bar{y}) + \epsilon u_m(x_2,\bar{y},t) + \ldots, \tag{5.3}$$

$$v = \epsilon^2 v_m(x_2,\bar{y},t) + \ldots. \tag{5.4}$$

Because derivatives involving time do not enter the equations until second-order terms are considered, we find that, apart from the time dependence, the structure of the main deck is similar to that which applies in the case of a steady inclined plate (see I). On substituting (5.2)–(5.4) into the full Navier–Stokes equations in moving co-ordinates and equating coefficients of the leading powers of ϵ to zero, we obtain

$$\left.\begin{aligned} p_m(x_2,\bar{y},t) &= p_m(x_2,0,t),\\ u_m(x_2,\bar{y},t) = A(x_2,t)\frac{\partial U_0}{\partial \bar{y}}, &\quad v_m(x_2,\bar{y},t) = -\frac{\partial A}{\partial x_2}(x_2,t)\,U_0(\bar{y}),\end{aligned}\right\} \tag{5.5}$$

where $A(x_2,t)$ is a function of x_2 and t to be determined and, from the match with the solution (4.12) upstream, satisfies

$$A(x_2,t) \sim (-i\alpha_0 S_0/4\lambda)\,(-x_2)^{-\frac{1}{2}}e^{it} \quad (x_2 \to -\infty). \tag{5.6}$$

From (4.2) we also have

$$p_m(x_2,0,t) \sim \tfrac{1}{2}\alpha_0 S_0^{\frac{3}{2}}(-x_2)^{\frac{1}{2}}e^{it} \quad (x_2 \to -\infty). \tag{5.7}$$

An equation relating p_m to A follows from the upper deck, where the appropriate solution has the form

$$p = \epsilon^2 p_u(x_2,\,Y,t)+\ldots, \tag{5.8}$$

$$u = 1+\epsilon^2 u_u(x_2,\,Y,t)+\ldots, \tag{5.9}$$

$$v = \epsilon^2 v_u(x_2,\,Y,t)+\ldots\,. \tag{5.10}$$

Again, no time derivatives are involved to first order and it may easily be shown that p_u+iv_u is a function of x_2+iY and t only, and that

$$p_u(x_2,0,t) = p_m(x_2,0,t), \quad v_u(x_2,0,t) = -\partial A(x_2,t)/\partial x_2. \tag{5.11}$$

The relations (5.11) are obtained from the match with the solution in the main deck as $Y \to 0$. It now follows that

$$p_m(x_2,0,t) = \frac{1}{\pi}\mathscr{F}\int_{-\infty}^{\infty}\frac{\partial A(x_2',t)/\partial x_2'\,dx_2'}{x_2-x_2'}. \tag{5.12}$$

Because of (5.7) here we use Hadamard's notion of the finite part (denoted by $\mathscr{F}$) of the infinite integral.

Finally we consider the lower deck, of thickness $O(\epsilon^5 l)$, where the solution has the form

$$p = \epsilon^2 p_l(x_2,t)+\ldots, \tag{5.13}$$

$$u = \epsilon u_l(x_2,z,t)+\ldots, \tag{5.14}$$

$$v = \epsilon^3 v_l(x_2,z,t)+\ldots, \tag{5.15}$$

and $p_l(x_2,t) = p_m(x_2,0,t)$. Here the first-order terms satisfy the conventional unsteady boundary-layer equations with the following boundary conditions for $x_2 < 0$:

$$u_l = v_l = 0 \quad \text{on} \quad z = 0, \tag{5.16}$$

$$u_l - \lambda z \to \lambda A(x_2,t) \quad \text{as} \quad z \to \infty, \tag{5.17}$$

$$u_l \to \lambda z \quad \text{as} \quad x_2 \to -\infty. \tag{5.18}$$

Condition (5.17) follows from the match with the solution (5.3) in the main deck whilst (5.18) represents the match with the sublayer flow (3.10) in the lower deck of the fore deck.

Before setting out the boundary conditions in $x_2 > 0$, it is convenient to consider the solution in $y < 0$; an analogous argument is used with a few changes in sign. The inviscid pressure and velocity perturbations which appear in (2.7), (3.5), (4.2) and (4.3) are of opposite sign. The only difference in the key equations (5.12) and (5.17) is that the sign of the term corresponding to $A(x_2, t)$ changes while the term corresponding to $p_m(x_2, 0, t)$ remains unaltered. If we denote the value of $p_l(x_2, t)$ by $p_T(x_2, t)$ when $y > 0$ and by $p_B(x_2, t)$ when $y < 0$ with a corresponding notation for $A_T(x_2, t)$ and $A_B(x_2, t)$, the fundamental problem in the lower deck for the oscillating plate can be stated as follows.

Solve

$$\tilde{S}_0 \frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{z}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}, \quad \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{z}} = 0, \tag{5.19}$$

with

$$\tilde{p}(\tilde{x}, t) = \begin{cases} \tilde{p}_T(\tilde{x}, t) = \dfrac{1}{\pi}\mathscr{F}\displaystyle\int_{-\infty}^{\infty} \frac{\partial \tilde{A}_T(x', t)/\partial x'\, dx'}{\tilde{x} - x'} \quad (\tilde{z} > 0), & (5.20) \\[2ex] \tilde{p}_B(\tilde{x}, t) = -\dfrac{1}{\pi}\mathscr{F}\displaystyle\int_{-\infty}^{\infty} \frac{\partial \tilde{A}_B(x', t)/\partial x'\, dx'}{\tilde{x} - x'} \quad (\tilde{z} < 0), & (5.21) \end{cases}$$

subject to the boundary conditions

$$\tilde{u} \to |\tilde{z}| \quad (\tilde{x} \to -\infty), \tag{5.22}$$

$$\tilde{u} = \tilde{v} = 0 \quad \text{on} \quad \tilde{z} = 0 \quad (\tilde{x} < 0), \tag{5.23}$$

$$\tilde{u} - \tilde{z} \to \tilde{A}_T(\tilde{x}, t) \quad (\tilde{z} \to \infty), \quad \tilde{u} + \tilde{z} \to -\tilde{A}_B(\tilde{x}, t) \quad (\tilde{z} \to -\infty), \tag{5.24}$$

$$\tilde{u}, \tilde{v} \text{ smooth for all } \tilde{z}, \quad \tilde{p}_T(\tilde{x}, t) = \tilde{p}_B(\tilde{x}, t) \quad (\tilde{x} > 0), \tag{5.25}$$

$$\tilde{p}_T \sim \tfrac{1}{2}\tilde{\alpha}_0 \tilde{S}_0^2 (-\tilde{x})^{\frac{1}{2}} e^{it}, \quad \tilde{p}_B \sim -\tfrac{1}{2}\tilde{\alpha}_0 \tilde{S}_0^2 (-\tilde{x})^{\frac{1}{2}} e^{it} \quad (\tilde{x} \to -\infty). \tag{5.26}$$

Here the constant λ has been eliminated from the problem by means of the transformations

$$\tilde{x} = \lambda^{\frac{5}{4}} x_2, \quad \tilde{z} = \lambda^{\frac{3}{4}} z, \quad \tilde{u} = \lambda^{-\frac{1}{4}} u_l, \quad \tilde{v} = \lambda^{-\frac{3}{4}} v_l, \quad \tilde{p} = \lambda^{-\frac{1}{2}} p_l, \tag{5.27}$$
$$\tilde{A} = \lambda^{\frac{3}{4}} A, \quad \tilde{S}_0 = \lambda^{-\frac{3}{2}} S_0, \quad \tilde{\alpha}_0 = \lambda^{\frac{15}{8}} \alpha_0.$$

Conditions (5.22)–(5.24) and (5.26) follow immediately from (5.16)–(5.18) and (5.7), whilst in the wake, condition (5.26) represents a continuous flow.

The general solution of the system (5.19)–(5.25) for this unsteady problem has not been attempted though recent advances have been made in the solution of two problems which were formulated earlier. Jobe (1973, private communication) has performed a numerical integration of the corresponding equations for the steady aligned flat plate studied by Stewartson (1969) and Messiter (1970), while Daniels (1974) has calculated the solution for the lifting case of I when the mainstream is supersonic. In the supersonic case the outer boundary condition (5.24) is slightly simplified because the upper-deck equation is the wave equation rather than the potential equation.

6. The solution in the lower deck as $|\tilde{x}| \to \infty$

In this section we give a brief description of the structure of the lower deck both as $\tilde{x} \to +\infty$ and as $\tilde{x} \to -\infty$. The purpose of this is to gain confidence in the consistency of the overall formulation of the problem.

(*a*) When $|\tilde{x}| \gg 1$ and $\tilde{x} < 0$ it is expected that the arguments given in I for the symmetric and antisymmetric parts of the pressure will be valid with a modification for the time dependence. The antisymmetric part of $\tilde{p}$, $\frac{1}{2}(\tilde{p}_T - \tilde{p}_B)$, which is zero when $\tilde{x} > 0$, will have an asymptotic expansion in descending powers of $(-\tilde{x})^{\frac{1}{2}}$, while the symmetric part $\frac{1}{2}(\tilde{p}_T + \tilde{p}_B)$ will be dominated by the term $-1{\cdot}7840/3^{\frac{3}{2}}(-\tilde{x})^{\frac{2}{3}}$ as is the pressure in the steady aligned problem. We therefore assume that for large negative $\tilde{x}$

$$\tilde{p}_T(\tilde{x}) = \tfrac{1}{2}\tilde{\alpha}_0 \tilde{S}_0^2(-\tilde{x})^{\frac{1}{2}} e^{it} + \tfrac{1}{2}\tilde{\alpha}_0 \frac{\tilde{S}_0^2 \tilde{C}_0}{(-\tilde{x})^{\frac{1}{2}}} e^{it} - \frac{1{\cdot}784}{3^{\frac{3}{2}}(-\tilde{x})^{\frac{2}{3}}} + O((-\tilde{x})^{-\frac{5}{6}}), \tag{6.1}$$

where

$$\tilde{C}_0 = \lambda^{\frac{5}{4}} C_0. \tag{6.2}$$

It emerges that there are two layers in this lower deck when $-\tilde{x} \gg 1$. When $\eta = \tilde{z}/3\,|2\tilde{x}|^{\frac{1}{3}}$ is $O(1)$ the velocity $\tilde{u}$ takes the form

$$\tilde{u} = \tilde{z} - \frac{i\tilde{\alpha}_0 \tilde{S}_0 e^{it}}{4(-\tilde{x})^{\frac{1}{2}}} + 1{\cdot}784 \frac{f'(\eta)}{-\tilde{x}} + O((-\tilde{x})^{-\frac{4}{3}}), \tag{6.3}$$

where f is the function of η satisfying $f(0) = f'(0) = 0$ and $f'(\infty) = 0{\cdot}183$ which occurs in the aligned problem (see Stewartson 1969).

However, when $\tilde{z}$ is $O(1)$ there is an inner layer which enables the boundary condition on $\tilde{u}$ at $\tilde{z} = 0$ to be satisfied. The solution in this layer is

$$\tilde{u} = \tilde{z} - \frac{i\tilde{\alpha}_0 \tilde{S}_0}{4(-\tilde{x})^{\frac{1}{2}}}[1 - \exp(-i^{\frac{1}{2}} \tilde{S}_0^{\frac{1}{2}} \tilde{z})]\, e^{it} + \frac{1{\cdot}784 f''(0)\, \tilde{z}}{3 \times 2^{\frac{1}{3}}(-\tilde{x})^{\frac{4}{3}}} + O((-\tilde{x})^{-\frac{5}{3}}). \tag{6.4}$$

It follows from (6.3) that, when $\tilde{x}$ is large and negative,

$$\tilde{A}_T(\tilde{x}, t) = -\frac{i\tilde{\alpha}_0 \tilde{S}_0 e^{it}}{4(-\tilde{x})^{\frac{1}{2}}} + \frac{0{\cdot}326}{-\tilde{x}} + O((-\tilde{x})^{-\frac{4}{3}}). \tag{6.5}$$

(*b*) When $|\tilde{x}| \gg 1$ and $\tilde{x} > 0$ the appropriate solution is that for Goldstein's (1930) inner wake, which is denoted by V_2 in figure 1, together with a time-dependent displacement of the centre-line. For large $\tilde{x}$ we write

$$\tilde{u} = \tfrac{1}{3}(\tfrac{1}{4}\tilde{x})^{\frac{1}{3}} \bar{g}_0'(\bar{\eta}) + \dots, \tag{6.6}$$

where

$$\bar{\eta} = [\tilde{z} - \mathcal{O}(\tilde{x}, t)]/3(2\tilde{x})^{\frac{1}{3}}. \tag{6.7}$$

Here $\bar{g}_0$ satisfies $\bar{g}_0''' + 2\bar{g}_0 \bar{g}_0'' - \bar{g}_0'^2 = 0$, $\bar{g}_0(0) = \bar{g}_0''(0) = 0$ and $g_0''(\infty) = 18$, and $\mathcal{O}(\tilde{x}, t)$ is defined by

$$\mathcal{O} = -\tfrac{1}{2}(\tilde{A}_T + \tilde{A}_B), \tag{6.8}$$

whose form for $\tilde{x} \gg 1$ we already know from (6.1) since in the upper deck $p_u + iv_u$ is a function of $x_2 + iY$ (and t) and $\partial A/\partial x_2 = -v_u(x_2, 0, t)$. It thus follows from (6.1) and the property that $\tilde{p}_T = \tilde{p}_B$ for $\tilde{x} > 0$ that for $\tilde{x} \gg 1$

$$\mathcal{O}(\tilde{x}, t) = -\tfrac{1}{3}\tilde{\alpha}_0 \tilde{S}_0^2 \tilde{x}^{\frac{3}{2}} e^{it} + \tilde{\alpha}_0 \tilde{S}_0^2 \tilde{C}_0 \tilde{x}^{\frac{1}{2}} e^{it} + \dots . \tag{6.9}$$

We may note that the equation $\tilde{z} = \mathscr{O}(\tilde{x}, t)$ of the centre-line of this wake is exactly the equation of the oscillating centre-streamline of the potential flow of §2 as it enters the trailing-edge region. This can be seen from (2.9), from which it follows that for $x^* - \frac{1}{2}l = O(\epsilon^3 l)$ the velocity on $y^* = 0$ is (U_∞, V_1), where

$$V_1 = -\tfrac{1}{2}U_\infty \epsilon^{\frac{1}{2}} \alpha_0 S_0^{\frac{2}{3}}\{(x^*/l - \tfrac{1}{2})^{\frac{1}{2}} - \epsilon^3 C_0 (x^*/l - \tfrac{1}{2})^{-\frac{1}{2}}\} e^{it}. \quad (6.10)$$

The asymptotic forms of the antisymmetric part of $\tilde{A}$ and the symmetric part of $\tilde{p}$ are derived from the properties of the Goldstein wake. Equation (6.6) implies that

$$\left.\begin{aligned} \tfrac{1}{2}(\tilde{A}_T(\tilde{x}, t) - \tilde{A}_B(\tilde{x}, t)) &= 1{\cdot}416\,(\tfrac{1}{4}\tilde{x})^{\frac{1}{3}} + \dots, \\ \tfrac{1}{2}(\tilde{p}_T(\tilde{x}, t) + \tilde{p}_B(\tilde{x}, t)) &= -1{\cdot}784/3^{\frac{2}{3}}\tilde{x}^{\frac{2}{3}} + \dots \end{aligned}\right\} \quad (6.11)$$

as $\tilde{x} \to \infty$, as in the steady aligned plate solution.

7. The linearized solution in the lower deck when $\tilde{S}_0 \gg 1$

An approximate solution of the fundamental problem (5.19)–(5.26) is possible if we linearize about the shear flow with which $\tilde{u}(\tilde{x}, \tilde{z}, t)$ merges at the outer edge of the lower deck in a similar way to that in I. We regard the resulting solution for $\tilde{u}$ as valid only in $\tilde{x} < 0$ since it is not expected that the linear shear is a good first approximation in the wake. However, the method of Wiener & Hopf enables the functions $\frac{1}{2}(\tilde{p}_T - \tilde{p}_B)$ and $\frac{1}{2}(\tilde{A}_T + \tilde{A}_B)$ to be determined for all $\tilde{x}$, the former vanishing for $\tilde{x} > 0$. An estimate for the Kutta constant C_0 may then be determined by comparison with the upstream form (2.7), which indicates that

$$\tfrac{1}{2}(\tilde{p}_T - \tilde{p}_B) \sim \tfrac{1}{2}\tilde{\alpha}_0 \tilde{S}_0^{\frac{2}{3}}\{(-\tilde{x})^{\frac{1}{2}} + \tilde{C}_0/(-\tilde{x})^{\frac{1}{2}}\} e^{it} \quad (\tilde{x} \to -\infty). \quad (7.1)$$

Denoting the values of $\tilde{u}(\tilde{x}, \tilde{z}, t)$ for $\tilde{z} > 0$ and $\tilde{z} < 0$ by $\tilde{u}_T(\tilde{x}, \tilde{z}, t)$ and $\tilde{u}_B(\tilde{x}, \tilde{z}, t)$ respectively, we write

$$\tilde{u}_T(\tilde{x}, \tilde{z}, t) = Z + \tilde{W}_T(\tilde{x}, Z)\, e^{it}, \quad \tilde{u}_B(\tilde{x}, \tilde{z}, t) = Z + \tilde{W}_B(\tilde{x}, Z)\, e^{it}, \quad (7.2)$$

where $Z = |\tilde{z}|$, in (5.19), neglect the nonlinear terms and subtract to obtain

$$i\tilde{S}_0(\tilde{W}_T - \tilde{W}_B) + Z\frac{\partial}{\partial \tilde{x}}(\tilde{W}_T - \tilde{W}_B) + (\tilde{V}_T + \tilde{V}_B) = -\frac{d}{d\tilde{x}}(\tilde{P}_T - \tilde{P}_B) + \frac{\partial^2}{\partial Z^2}(\tilde{W}_T - \tilde{W}_B), \quad (7.3)$$

$$\partial(\tilde{W}_T - \tilde{W}_B)/\partial \tilde{x} + \partial(\tilde{V}_T + \tilde{V}_B)/\partial Z = 0. \quad (7.4)$$

Here $\tilde{p}_T = \tilde{P}_T(\tilde{x})\, e^{it}$ and $\tilde{v}_T = \tilde{V}_T(\tilde{x}, Z)\, e^{it}$ with corresponding notation for $\tilde{P}_B$ and $\tilde{V}_B$. Differentiation of (7.3) with respect to Z and elimination of $\tilde{V}_T + \tilde{V}_B$ using (7.4) gives the fundamental equation to be considered for $\tilde{x} < 0$ only as

$$i\tilde{S}_0 \frac{\partial W}{\partial Z} + Z\frac{\partial^2 W}{\partial \tilde{x}\, \partial Z} = \frac{\partial^3 W}{\partial Z^3} \quad (\tilde{x} < 0), \quad (7.5)$$

where $W = \frac{1}{2}(\tilde{W}_T - \tilde{W}_B)$; this is to be solved subject to the conditions

$$\partial^2 W/\partial Z^2 = Q(\tilde{x}) \quad (Z = 0); \quad W \to \tfrac{1}{2}(\tilde{\tilde{A}}_T + \tilde{\tilde{A}}_B) \quad (Z \to \infty). \quad (7.6a, b)$$

Here $Q(\tilde{x}) = \frac{1}{2}d(\tilde{P}_T - \tilde{P}_B)/d\tilde{x}$ and $\tilde{A}_{T,B}(\tilde{x}, t) = \tilde{\tilde{A}}_{T,B}(\tilde{x})\, e^{it}$. The boundary condition as $Z \to \infty$ follows from (5.24).

We now define the Fourier transform of $W(\tilde{x},Z)$ as $\overline{W}(\omega,Z)$, where

$$\overline{W}(\omega,Z) = \int_{-\infty}^{\infty} W(\tilde{x},Z)\,e^{-i\omega\tilde{x}}d\tilde{x}, \tag{7.7}$$

and transform (7.5) to obtain the solution

$$\frac{\partial\overline{W}}{\partial Z}\frac{\partial Z}{\partial Z_1} = \frac{\overline{Q}_+(\omega)\,\mathrm{Ai}\,(Z_1)\,e^{-\frac{1}{3}i\pi}}{(\omega-i\delta)^{\frac{2}{3}}\,\mathrm{Ai}'\,(Z_0)} + M_-(\omega,Z_1), \tag{7.8}$$

where

$$Z_1 = \frac{e^{\frac{1}{6}i\pi}}{(\omega-i\delta)^{\frac{2}{3}}}(\tilde{S}_0+\omega Z), \quad Z_0 = \frac{e^{\frac{1}{6}i\pi}\tilde{S}_0}{(\omega-i\delta)^{\frac{2}{3}}}. \tag{7.9}$$

$\overline{Q}_+(\omega)$ is the Fourier transform of $Q(\tilde{x})$; the suffix + indicates that it is a regular function of the complex variable ω for $\mathrm{Im}\,\omega > 0$, since we require that $Q(\tilde{x}) \equiv 0$ for $\tilde{x} > 0$. The solution (7.8) satisfies the boundary condition (5.23) on $Z = 0$ for $\tilde{x} < 0$ and contains the additional function $M_-(\omega, Z_1)$, regular for $\mathrm{Im}\,\omega < 0$, as the equation and boundary conditions satisfied by $W(\tilde{x},Z)$ for $\tilde{x} > 0$ are unspecified. The parameter δ is introduced for convenience and the limit $\delta \to 0+$ will eventually be taken.

Now we define $\overline{C}(\omega)$ as the Fourier transform of $\frac{1}{2}d^2(\tilde{\tilde{A}}_T+\tilde{\tilde{A}}_B)/d\tilde{x}^2$ and, using the fact that both v_u and p_u are harmonic in the variables x_2 and Y in the upper deck, we find, as in I, that

$$-|\omega|\,\overline{Q}_+(\omega) = i\omega\overline{C}(\omega). \tag{7.10}$$

Transformation of boundary condition (7.6*b*) gives

$$-\omega^2\overline{W}(\omega,Z) \to \overline{C}(\omega) \quad \text{as} \quad Z \to \infty, \tag{7.11}$$

and since $\overline{W}(\omega, 0) = 0$ we have

$$-\frac{\overline{C}(\omega)}{\omega^2} = \int_{Z_0}^{\infty} \frac{\partial\overline{W}}{\partial Z}\frac{\partial Z}{\partial Z_1}dZ_1. \tag{7.12}$$

The complicated form of the integral (7.12), with integrand (7.8), leads to a formidable factorization problem in the subsequent Wiener–Hopf procedure unless $\tilde{S}_0$ is either small or large. The steady lifting case of I corresponded to the situation in which $\tilde{S}_0 = 0$ and in order to make further progress in the problem of the rapidly oscillating aerofoil we now assume that $\tilde{S}_0 \gg 1$ so that we may replace the Airy functions in (7.8) by their asymptotic expansions. We then obtain

$$-\frac{\overline{C}(\omega)}{\omega^2} = -\frac{e^{-\frac{1}{2}i\pi}\overline{Q}_+(\omega)}{\tilde{S}_0} + N_-(\omega) \quad (\tilde{S}_0 \gg 1), \tag{7.13}$$

where

$$N_-(\omega) = \int_{Z_0}^{\infty} M_-(\omega,Z_1)\,dZ_1.$$

The transform $\overline{C}(\omega)$ may now be eliminated between (7.10) and (7.13) and replacing $|\omega|$ by $(\omega-i\delta)^{\frac{1}{2}}(\omega+i\delta)^{\frac{1}{2}}$ we have

$$\overline{Q}_+(\omega)\,K_+(\omega) = i\tilde{S}_0 N_-(\omega)\,K_-(\omega), \tag{7.14}$$

where $$K_+(\omega)/K_-(\omega) = K(\omega) = 1+\tilde{S}_0(\omega+i\delta)^{\frac{1}{2}}/(\omega-i\delta)^{\frac{5}{2}}. \tag{7.15}$$

Here the left-hand side of (7.14) is regular for $\operatorname{Im}\omega > 0$ and the right-hand side for $\operatorname{Im}\omega < 0$ on the assumption that the factorization (7.15) has been made. We make the additional assumption, which may be justified *a posteriori*, that these regions of regularity may be extended to $\operatorname{Im}\omega > -\delta$ and $\operatorname{Im}\omega < \delta$ respectively; the two sides are then equal and regular on a dense set of points, and so, by analytic continuation, define a function which is regular everywhere.

The function $K(\omega)$ is regular in the ω plane cut along the positive imaginary axis from $+i\delta$ to $+i\infty$ and along the negative imaginary axis from $-i\delta$ to $-i\infty$ and has zeros at $\omega = -\tilde{S}_0^{\frac{1}{2}}+\frac{3}{2}i\delta$ and $\omega = \pm i\tilde{S}_0^{\frac{1}{2}}+\frac{3}{2}i\delta$. Thus $\log K(\omega)$ is regular and non-zero in the strip $-\delta < \operatorname{Im}\omega < \delta$, $-\infty < \operatorname{Re}\omega < \infty$ and tends to zero as $\operatorname{Re}\omega \to \pm\infty$; the factorization (7.15) is thus possible and is performed in the usual way (see, *for example*, Noble 1958) to give

$$K_-(\omega) = -\frac{(\omega-i\delta)^{\frac{5}{2}}}{\tilde{S}_0^{\frac{1}{2}}(\tilde{S}_0^{\frac{1}{2}}-\frac{3}{2}i\delta+\omega)\,(i\tilde{S}_0^{\frac{1}{2}}+\frac{3}{2}i\delta-\omega)^{\frac{1}{2}}}\exp\left\{-\frac{2\tilde{S}_0}{\pi i}\int_0^\infty \frac{\sigma\log(\sigma+i\omega)\,d\sigma}{\sigma^4-\tilde{S}_0^2}\right\}, \tag{7.16}$$

$$K_+(\omega) = \frac{i}{\tilde{S}_0^{\frac{1}{2}}}(\omega i\delta+)^{\frac{1}{2}}\,(i\tilde{S}_0^{\frac{1}{2}}-\tfrac{3}{2}i\delta+\omega)^{\frac{1}{2}}\exp\left\{-\frac{2\tilde{S}_0}{\pi i}\int_0^\infty \frac{\sigma\log(\sigma-i\omega)\,d\sigma}{\sigma^4-\tilde{S}_0^2}\right\}. \tag{7.17}$$

The arbitrary multiplicative factor has been chosen so that

$$\frac{K_\pm(\omega)}{i\omega} \to \tilde{S}_0^{-\frac{1}{2}} \quad (\operatorname{Re}\omega\to\infty), \tag{7.18}$$

and the integrals in (7.16), (7.17) and (7.20)–(7.23) below are to be interpreted as principal-value integrals.

We now return to (7.14) and set both sides equal to a constant D, this being the function, regular everywhere, which is appropriate to the limiting behaviour of $\frac{1}{2}(\tilde{P}_T-\tilde{P}_B)$ as $\tilde{x}\to-\infty$ given by (5.26). Thus

$$Q(\tilde{x}) = \frac{D}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega\tilde{x}}\,d\omega}{K_+(\omega)}, \tag{7.19}$$

and given $\bar{Q}_+(\omega)$, $\bar{C}(\omega)$ follows from (7.10) and $[\partial\overline{W}/\partial Z]_{Z=0}$ from (7.8). With use of the substitutions $\omega = -i\tilde{S}_0^{\frac{1}{2}}r$ and $\omega = i\tilde{S}_0^{\frac{1}{2}}s$ where necessary, the three Fourier transforms are inverted to give.

$$\frac{1}{2}\frac{d}{d\tilde{x}}(\tilde{P}_T-\tilde{P}_B) = \begin{cases} -\dfrac{\tilde{S}_0^{\frac{1}{2}}D}{\pi}\displaystyle\int_0^\infty \frac{\exp(\tilde{S}_0^{\frac{1}{2}}\tilde{x}r)\,(1-ir)\,(1+r)^{\frac{1}{2}}\exp\{I(r)\}\,dr}{r^{\frac{1}{2}}(1-r^4)} \\ \qquad\qquad -\dfrac{D\tilde{S}_0^{\frac{1}{2}}}{2}\exp(\tilde{S}_0^{\frac{1}{2}}\tilde{x})\exp(\tfrac{3}{16}i\pi) & (\tilde{x}<0), \\ 0 & (\tilde{x}>0), \end{cases} \tag{7.20}$$

$$\frac{1}{2}\frac{d^2}{d\tilde{x}^2}(\tilde{\tilde{A}}_T+\tilde{\tilde{A}}_B) = \begin{cases} -\dfrac{\tilde{S}_0^{\frac{1}{2}}Di}{\pi}\displaystyle\int_0^\infty \frac{\exp(\tilde{S}_0^{\frac{1}{2}}\tilde{x}r)\,(1-ir)\,(1+r)^{\frac{1}{2}}r^{\frac{3}{2}}\exp\{I(r)\}\,dr}{1-r^4} \\ \qquad\qquad -\dfrac{D\tilde{S}_0^{\frac{1}{2}}}{2}\exp(\tilde{S}_0^{\frac{1}{2}}\tilde{x})\exp(\tfrac{11}{16}i\pi) & (\tilde{x}<0), \\ \dfrac{\tilde{S}_0^{\frac{1}{2}}D}{\pi}\displaystyle\int_0^\infty \frac{\exp(-\tilde{S}_0^{\frac{1}{2}}\tilde{x}s)\exp\{I(s)\}\,ds}{s^{\frac{1}{2}}(1+s)^{\frac{1}{2}}} & (\tilde{x}>0), \end{cases} \tag{7.21}$$

$$\left.\frac{\partial W}{\partial Z}\right|_{Z=0} = \frac{\exp(-\frac{1}{4}i\pi)\,D}{\pi}\int_0^\infty \frac{\exp(\tilde{S}_0^{\frac{1}{2}}\tilde{x}r)\,(1-ir)\,(1+r)^{\frac{1}{2}}\exp\{I(r)\}\,dr}{r^{\frac{1}{2}}(1-r^4)}$$

$$-\frac{D\exp(-\frac{1}{16}i\pi)}{2}\exp(\tilde{S}_0^{\frac{1}{2}}\tilde{x}) \quad (\tilde{x} < 0), \qquad (7.22)$$

where

$$I(r) = \frac{2}{\pi i}\int_0^\infty \frac{\sigma\log(\sigma+r)\,d\sigma}{\sigma^4-1}. \qquad (7.23)$$

The constant D is determined by the boundary condition (5.26), which requires

$$(-\tilde{x})^{\frac{1}{2}}d(\tilde{P}_T-\tilde{P}_B)/d\tilde{x} \to -\tfrac{1}{2}\tilde{\alpha}_0\tilde{S}_0^2 \quad (\tilde{x}\to-\infty). \qquad (7.24)$$

The forms of the integrals (7.20)–(7.22) may be found for large and small values of $|\tilde{x}|$ once the asymptotic expansions of the function $I(r)$ are known for small and large values of r respectively. Since, from (7.23), $I(r)+I(1/r) = -\frac{1}{8}i\pi$, we need only consider small values of r, and we find that

$$I(r) \sim -\tfrac{1}{8}i\pi + \tfrac{1}{2}ir + (i/\pi)\,r^2(\log r - \tfrac{1}{2}) + O(r^3) \quad (r\to 0). \qquad (7.25)$$

Comparison of (7.20) and (7.24) then determines the constant D as

$$D = \tfrac{1}{4}\tilde{\alpha}_0\tilde{S}_0^{\frac{7}{4}}e^{\frac{1}{8}i\pi}\pi^{\frac{1}{2}}, \qquad (7.26)$$

and when $\tilde{x}$ is large and negative we have

$$\tfrac{1}{2}(\tilde{P}_T-\tilde{P}_B) = \tfrac{1}{2}\tilde{\alpha}_0\tilde{S}_0^2(-\tilde{x})^{\frac{1}{2}} - \frac{\tilde{\alpha}_0\tilde{S}_0^{\frac{3}{2}}(1-i)}{8(-\tilde{x})^{\frac{1}{2}}} + O\left(\frac{\log(-\tilde{x})}{(-\tilde{x})^{\frac{3}{2}}}\right), \qquad (7.27)$$

$$\tfrac{1}{2}(\tilde{\tilde{A}}_T+\tilde{\tilde{A}}_B) = -\frac{i\tilde{\alpha}_0\tilde{S}_0}{4(-\tilde{x})^{\frac{1}{2}}} - \frac{(1+i)\,\tilde{\alpha}_0\tilde{S}_0^{\frac{1}{2}}}{16(-\tilde{x})^{\frac{3}{2}}} + O\left(\frac{\log(-\tilde{x})}{(-\tilde{x})^{\frac{5}{2}}}\right), \qquad (7.28)$$

$$\frac{1}{2}\left\{\left(\frac{\partial\tilde{u}}{\partial\tilde{z}}\right)_T + \left(\frac{\partial\tilde{u}}{\partial\tilde{z}}\right)_B\right\}_{\tilde{z}=0} = \frac{(1-i)\,\tilde{\alpha}_0\tilde{S}_0^{\frac{3}{4}}e^{it}}{2^{\frac{3}{2}}(-\tilde{x})^{\frac{1}{2}}} + \frac{2^{\frac{1}{2}}i\tilde{\alpha}_0\tilde{S}_0e^{it}}{16(-\tilde{x})^{\frac{3}{2}}} + O\left(\frac{\log(-\tilde{x})}{(-\tilde{x})^{\frac{5}{2}}}\right) \qquad (7.29)$$

while, for $\tilde{x} > 0$,

$$\tfrac{1}{2}(\tilde{\tilde{A}}_T+\tilde{\tilde{A}}_B) = \tfrac{1}{3}\tilde{\alpha}_0\tilde{S}_0^2\tilde{x}^{\frac{3}{2}} + \tfrac{1}{4}\tilde{\alpha}_0\tilde{S}_0^{\frac{3}{2}}(1-i)\,\tilde{x}^{\frac{1}{2}} + O(1). \qquad (7.30)$$

The leading term of (7.29) is seen to be consistent with the upstream sublayer solution (3.10). Of the four arbitrary constants which arise from the integration of (7.21), two are chosen to satisfy the required asymptotic behaviour of $\frac{1}{2}(\tilde{A}_T+\tilde{A}_B)$ as $\tilde{x}\to-\infty$, which is given by (5.6), whilst the other two ensure that $\frac{1}{2}(\tilde{A}_T+\tilde{A}_B)$ and $\frac{1}{2}d(\tilde{A}_T+\tilde{A}_B)/d\tilde{x}$ are continuous at the trailing edge. The function $\frac{1}{2}d^2(\tilde{A}_T+\tilde{A}_B)/d\tilde{x}^2$, on the other hand, is not finite at the trailing edge and we have

$$\frac{1}{2}\frac{d^2}{\partial\tilde{x}^2}(\tilde{A}_T+\tilde{A}_B) \sim -\frac{\tilde{\alpha}_0\tilde{S}_0^{\frac{9}{4}}e^{it+\frac{1}{8}i\pi}}{4\pi^{\frac{1}{2}}}\log\tilde{x} \quad (\tilde{x}\to 0+). \qquad (7.31)$$

The arbitrary constant in the integration of (7.20) is determined by the requirement that the pressure be continuous at $\tilde{x} = 0$.

We may now compare the linearized solution (7.27) with (7.1) to give an estimate of the previously unknown constant $\tilde{C}_0$ as

$$\tilde{C}_0 = -(1-i)/4\tilde{S}_0^{\frac{1}{2}}. \qquad (7.32)$$

8. Extension to include a plunging mode

With suitable adaptation of the outer potential flow over the plate, it may easily be shown that the results of §§3–7, describing the trailing-edge flow of a plate in pitching motion, may also be applied to the case of a plate performing harmonic oscillations in a direction perpendicular to the stream at infinity (or to a combination of both). In order to contrast the two effects we shall now consider the latter in isolation. For such a plunging motion, the equation of the plate may be written as

$$y^* = -\tfrac{1}{2}h^*l\exp(i\omega^*t^*) \quad (-\tfrac{1}{2}l \leqslant x^* \leqslant \tfrac{1}{2}l), \tag{8.1}$$

where $\omega^* = O(\epsilon^{-2})$ is the frequency (as before) and, for convenience, we write the amplitude as $\frac{1}{2}h^*l = O(\epsilon^{\frac{9}{2}}l)$. The functions $a_n(t^*)$ $(n = 0, 1, 2)$ of (2.2) then become

$$a_0(t^*) = -\tfrac{1}{2}ih^*(\omega^*l/U_\infty)\exp(i\omega^*t^*)+O(h^*) \quad \text{as} \quad \omega^*\to\infty, \tag{8.2}$$

$$a_1(t^*) = \tfrac{1}{4}h^*(\omega^*l/U_\infty)^2\exp(i\omega^*t^*), \quad a_2(t^*) = 0, \tag{8.3}$$

and we find that the potential-flow pressure on the upper surface of the plate is

$$\frac{p^*-p_\infty}{\rho U_\infty^2} = \tfrac{1}{2}h^*S^2\left\{\left(\frac{1}{2}-\frac{x^*}{l}\right)^{\frac{1}{2}}+C\left(\frac{1}{2}-\frac{x^*}{l}\right)^{-\frac{1}{2}}\right\}\exp(i\omega^*t^*), \tag{8.4}$$

when $\frac{1}{2}-x^*/l \ll 1$. Here $S = \omega^*l/U_\infty$ as before and C is defined by the formula (2.5) with α^* replaced by h^*. Comparison of the result (8.4) with the corresponding formula (2.7) now reveals that all the formulae obtained in §§3–7 apply to the plate performing the plunging mode (8.1), given that α^* is replaced by h^* and that the transformations (3.1) and (3.2) are replaced by

$$x^*/l-\tfrac{1}{2} = x, \quad y^*/l = y-\tfrac{1}{2}h^*e^{it}, \tag{8.5}$$

$$u^* = U_\infty u, \quad v^* = U_\infty(v-\tfrac{1}{2}ih^*S\,e^{it}). \tag{8.6}$$

The effect of the additional terms in the equations of motion is again of sufficiently lower order to be neglected in the boundary layer and triple deck. In general, then, the trailing-edge effect of a pitching motion is seen to be equivalent to that of a plunging motion of half the amplitude.

9. Results and discussion

The investigation undertaken in the previous sections has led to a seemingly consistent picture of the overall field though admittedly for limiting values of the parameters involved. The amplitude α^*l of the oscillation, which is taken to be $O(\epsilon^{\frac{9}{2}}l)$, is so small that, to the order considered, the additional terms in the Navier–Stokes equations due to the change to rotating axes are negligible in the boundary layer and triple deck. The frequency parameter S is written as $S = S_0/\epsilon^2$, where S_0, although independent of Reynolds number, is taken to be large for the purposes of the linearized solution for the sublayer in §7. The Reynolds number R is related to ϵ by $R = \epsilon^{-8}$.

The principal results may be summarized as follows. We have considered a rapidly oscillating aerofoil of length l, in a stream U_∞, whose displacement at time t^* is given by

$$y^* = -2\alpha^* x^* \cos\omega^* t^* \quad (-\tfrac{1}{2}l \leqslant x^* \leqslant \tfrac{1}{2}l), \tag{9.1}$$

with $\alpha^* = O(\epsilon^{\frac{5}{2}})$ and $\omega^* l/U_\infty = O(\epsilon^{-2})$. The potential flow (2.7) around this aerofoil contains an arbitrary constant C which is Reynolds number dependent and tends to zero as $R \to \infty$ to satisfy the Kutta–Joukowski hypothesis of zero loading at the trailing edge in the inviscid limit. By matching the outer flow to the flow in the triple deck we have shown that $C = O(\epsilon^3)$ and have made an estimate of its value. The outcome is that, when $S_0 \gg 1$, the potential-flow pressure is given by (2.7) as

$$\frac{p^* - p_\infty}{\rho U_\infty^2} = \tfrac{1}{2}\epsilon^{\frac{1}{2}}\alpha_0 S_0^2 \left(\frac{1}{2} - \frac{x^*}{l}\right)^{\frac{1}{2}} \cos\omega^* t^* - \frac{2^{\frac{1}{2}}}{8}\epsilon^{\frac{7}{2}} \frac{\alpha_0 S_0^{\frac{3}{2}}}{\lambda^{\frac{1}{2}}} \left(\frac{1}{2} - \frac{x^*}{l}\right)^{-\frac{1}{2}} \cos\left(\omega^* t^* - \frac{\pi}{4}\right). \tag{9.2}$$

The overall displacement-effect modification to $(p^* - p_\infty)/\rho U_\infty^2$ would be

$$O(\alpha^* (\omega^* l/U_\infty)^2 \epsilon^4)$$

and thus smaller by a factor ϵ than the trailing-edge effect given in (9.2). The pressure on the plate is also given by (9.2) except in the triple deck, where $(p^* - p_\infty)/\rho U_\infty^2$ is $O(\epsilon^2)$ and the singularity of (9.2) will be smoothed out. However the streamwise extent of the deck is $O(\epsilon^3)$ so the contribution to the lift and pitching moment from this region will be $O(\epsilon^5)$ and thus smaller than that from the rest of the plate by a factor $\epsilon^{\frac{3}{2}}$. The lift and pitching moment on the aerofoil calculated from this potential flow are, from (2.6),

$$\frac{L}{\rho l U_\infty^2} = -\tfrac{3}{4}\epsilon^{\frac{5}{2}}\pi\alpha_0 S_0 \sin\omega^* t^* + \frac{2^{\frac{1}{2}}}{8}\epsilon^{\frac{7}{2}} \frac{\pi\alpha_0 S_0^{\frac{3}{2}}}{\lambda^{\frac{1}{2}}} \cos\left(\omega^* t^* - \frac{\pi}{4}\right), \tag{9.3}$$

$$\frac{M}{\rho l^2 U_\infty^2} = -\tfrac{1}{64}\epsilon^{\frac{1}{2}}\pi\alpha_0 S_0^2 \cos\omega^* t^* - \tfrac{1}{16}\epsilon^{\frac{5}{2}}\pi\alpha_0 S_0 \sin\omega^* t^* + \frac{2^{\frac{1}{2}}}{32}\epsilon^{\frac{7}{2}} \frac{\pi\alpha_0 S_0^{\frac{3}{2}}}{\lambda^{\frac{1}{2}}} \cos\left(\omega^* t^* - \frac{\pi}{4}\right), \tag{9.4}$$

where the terms neglected are $O(\epsilon^{\frac{9}{2}})$.

When $x^* - \frac{1}{2}l = O(l)$ the streamline leaving the trailing edge has, from (2.9), the quation

$$y^* = -\tfrac{1}{2}\epsilon^{\frac{5}{2}} l\alpha_0 S_0 \left(\frac{x^*}{l} - \frac{1}{2}\right)^{\frac{1}{2}} \sin\omega^* t^* + \frac{2^{\frac{1}{2}}}{8}\epsilon^{\frac{11}{2}} \frac{l\alpha_0 S_0^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}} \left(\frac{x^*}{l} - \frac{1}{2}\right)^{-\frac{1}{2}} \cos\left(\omega^* t^* + \frac{\pi}{4}\right), \tag{9.5}$$

with a phase lag of $\frac{1}{2}\pi$ behind the plate, though when $x^* - \frac{1}{2}l = O(\epsilon^3 l)$ it emerges from the triple deck in the form

$$y^* = -\tfrac{1}{3}\epsilon^{\frac{1}{2}} l\alpha_0 S_0^2 \left(\frac{x^*}{l} - \frac{1}{2}\right)^{\frac{3}{2}} \cos\omega^* t^* - \frac{2^{\frac{1}{2}}}{4}\epsilon^{\frac{7}{2}} \frac{l\alpha_0 S_0^{\frac{3}{2}}}{\lambda^{\frac{1}{2}}} \left(\frac{x^*}{l} - \frac{1}{2}\right)^{\frac{1}{2}} \cos\left(\omega^* t^* - \frac{\pi}{4}\right). \tag{9.6}$$

The value of the potential-flow velocity on the top side of the plate may be obtained from (2.8), and that on the lower side by changing the sign of α^*. When $\frac{1}{2}l - x^* = O(l)$ it is

$$U_1 = U_\infty + \tfrac{1}{4}\epsilon^{\frac{5}{2}} U_\infty \alpha_0 S_0 \left(\frac{1}{2} - \frac{x^*}{l}\right)^{-\frac{1}{2}} \sin\omega^* t^* - \frac{2^{\frac{1}{2}}}{16}\epsilon^{\frac{11}{2}} \frac{U_\infty \alpha_0 S_0^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}} \left(\frac{1}{2} - \frac{x^*}{l}\right)^{-\frac{3}{2}} \cos\left(\omega^* t^* + \frac{\pi}{4}\right) \tag{9.7}$$

but when $\frac{1}{2}l - x^* = O(\epsilon^3 l)$ it is given by

$$U_1 = U_\infty - \tfrac{1}{2}\epsilon^{\frac{1}{2}} U_\infty \alpha_0 S_0^2 \left(\frac{1}{2} - \frac{x^*}{l}\right)^{\frac{1}{2}} \cos \omega^* t^* + \frac{2^{\frac{1}{2}}}{8} \epsilon^{\frac{7}{2}} \frac{U_\infty \alpha_0 S_0^{\frac{3}{2}}}{\lambda^{\frac{1}{2}}} \left(\frac{1}{2} - \frac{x^*}{l}\right)^{-\frac{1}{2}} \cos\left(\omega^* t^* - \frac{\pi}{4}\right). \tag{9.8}$$

For the plunging mode in which the equation of the aerofoil is

$$y^* = -\tfrac{1}{2} h^* l \cos \omega^* t^* \quad (-\tfrac{1}{2}l \leqslant x^* \leqslant \tfrac{1}{2}l) \tag{9.9}$$

(9.2) holds with α^* replaced by h^* as do (9.5)–(9.8). The overall effects of the plunging mode, however, are quite different from those of the pitching mode, for (9.3) and (9.4) must be replaced by

$$\frac{L}{\rho l U_\infty^2} = -\tfrac{1}{8}\pi \epsilon^{\frac{1}{2}} h_0 S_0^2 \cos \omega^* t^* - \tfrac{1}{4}\pi \epsilon^{\frac{5}{2}} h_0 S_0 \sin \omega^* t^* + \frac{2^{\frac{1}{2}}}{8} \pi \epsilon^{\frac{7}{2}} \frac{h_0 S_0^{\frac{3}{2}}}{\lambda^{\frac{1}{2}}} \cos\left(\omega^* t^* - \frac{\pi}{4}\right), \tag{9.10}$$

$$\frac{M}{\rho l^2 U_\infty^2} = \tfrac{1}{16}\pi \epsilon^{\frac{5}{2}} h_0 S_0 \sin \omega^* t^* + \frac{2^{\frac{1}{2}}}{32} \pi \epsilon^{\frac{7}{2}} \frac{h_0 S_0^{\frac{3}{2}}}{\lambda^{\frac{1}{2}}} \cos\left(\omega^* t^* - \frac{\pi}{4}\right), \tag{9.11}$$

where h_0 corresponds to the scaled parameter α_0 and is defined by $h^* = \epsilon^{\frac{9}{2}} h_0$. A comparison of these results with (9.3) and (9.4) shows that, as intuition would suggest, the plunging motion generates a greater lift, whilst the pitching motion has a larger moment about the mid-chord of the plate.

We are grateful to Professor G. J. Hancock and Professor K. Stewartson for their interest in this work and for their helpful discussions.

REFERENCES

Blasius, H. 1908 Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z. angew. Math. Phys.* **56**, 1–37.

Brown, S. N. & Stewartson, K. 1970 Trailing-edge stall. *J. Fluid Mech.* **42**, 561–584.

Daniels, P. G. 1974 Numerical and asymptotic solutions for the supersonic flow near the trailing edge of a flat plate at incidence. *J. Fluid Mech.* **63**, 641–656.

Goldstein, S. 1930 Concerning some solutions of the boundary-layer equations in hydrodynamics. *Proc. Camb. Phil. Soc.* **26**, 1–30.

Messiter, A. F. 1970 Boundary-layer flow near the trailing edge of a flat plate. *S.I.A.M. J. Appl. Math.* **18**, 241–257.

Noble, B. 1958 *The Wiener–Hopf Technique*. Pergamon.

Orszag, S. A. & Crow, S. C. 1970 Instability of a vortex sheet leaving a semi-infinite plate. *Studies in Appl. Math.* **49**, 167–181.

Pfizenmaier, E. & Bechert, D. 1973 Optimal compensation measurements on the non-stationary exit condition at a nozzle discharge edge. *Dtsche Luft- & Raumfahrt*, FB 73–93.

Riley, N. 1974 Flows with concentrated vorticity: a report on EUROMECH 41. *J. Fluid Mech.* **62**, 33–39.

Robinson, A. & Laurmann, J. A. 1956 *Wing Theory*. Cambridge University Press.

Shen, S. F. & Crimi, P. 1965 The theory for an oscillating thin aerofoil as derived from the Oseen equations. *J. Fluid Mech.* **23**, 585–609.

Stewartson, K. 1969 On the flow near the trailing edge of a flat plate, II. *Mathematika*, **16**, 106–121.

Van de Vooren, A. I. & Van de Vel, H. 1964 Unsteady profile theory in incompressible flow. *Arch. Mech. Stosowanej.* **16**, 709–735.